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# A Probabilistic Approach to the Equation $Lu = -u^2$

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Let $L$ be a second order elliptic differential operator and let $D$ be an arbitrary open subset of $\mathbb{R}^d$. In [1] we introduced a class $\mathscr{U}_1(D)$ of positive solutions of the [illegible] characterization of $\mathscr{U}_1(D)$ and a probabilistic representation of $u \in \mathscr{U}_1(D)$ in terms of a superdiffusion. Similar results are obtained also for a parabolic equation $\dot{u} + Lu = -u^2$. © 2000 Academic Press



## 1. INTRODUCTION

1.1. *Classes $\mathscr{U}_1(D)$ and $\mathscr{H}_1(D)$.* Let $L$ be a second order uniformly elliptic differential operator in $\mathbb{R}^d$ with bounded smooth coefficients such that $L1 = 0$. In [1], we investigated positive solutions of the equation $Lu = -\psi(u)$ for a wide class of positive functions $u$. Now we restrict ourselves by the case $\psi(u) = u^2$. Denote by $\mathscr{U}(D)$ the set of all positive solutions of the equation

$$Lu = -u^2 \qquad \text{in } D \tag{1.1}$$

and let $\mathscr{H}(D)$ stand for the class of all positive solutions of the equation $Lh = 0$ (we call them $L$-harmonic functions). We consider a diffusion $\xi = (\xi_t, \Pi_x)$ in $\mathbb{R}^d$ with the generator $L$ and we introduce the Green operator

$$G_D f(x) = \Pi_x \int_0^\tau f(\xi_s)\, ds \tag{1.2}$$

and the Poisson operator

$$K_D f(x) = \Pi_x f(\xi_\tau), \tag{1.3}$$



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where $\tau$ is the first exit time of $\xi$ from $F$. (Both operators act on the space $\mathscr{B}_+$ of a Borel functions on $\mathbb{R}^d$ with values in $[0, \infty]$.) The differential Eq. (1.1) is closely related to an integral equation,

$$u = \mathscr{E}_D(u) + h, \tag{1.4}$$

where $\mathscr{E}_D(f) = G_D(f^2)$ and $h \in \mathscr{H}(D)$.

We proved in [1] that:

1.1.A. For every $u \in \mathscr{U}(D)$ there exists a maximal $h \in \mathscr{H}(D)$ dominated by $u$. We denote it $j_D(u)$.

1.1.B. For every $h \in \mathscr{H}(D)$ there is a minimal solution of Eq. (1.4). We denote it $i_D(h)$. We have:

1.1.B.1. If $\varphi_1 \leqslant \varphi_2$ and $D_1 \subset D_2$, then $i_{D_1}(\varphi_1) \leqslant i_{D_2}(\varphi_2)$.

1.1.B.2. If $\varphi_n \uparrow \varphi$ and $D_n \uparrow D$, then $i_{D_n}(\varphi_n) \uparrow i_D(\varphi)$.

1.1.C. If $h \in \mathscr{H}(D)$ and $i_D(h) < \infty$ at some point $x \in D$, then

$$j_D[i_D(h)] = h.$$

1.1.D. For every $u \in \mathscr{U}(D)$,

$$i_D[j_D(u)] \leqslant u.$$

We denote as $\mathscr{H}_1(D)$ the class of all $h \in \mathscr{H}(D)$ such that $i_D(h)$ is locally bounded. We put $u \in \mathscr{U}_1(D)$ if $u \in \mathscr{U}(D)$ and $i_D[j_D(u)] = u$.

1.1.E. $j_D$ is a 1–1 mapping from $\mathscr{U}_1(D)$ onto $\mathscr{H}_1(D)$ and $i_D$ is the inverse mapping from $\mathscr{H}_1(D)$ onto $\mathscr{U}_1(D)$.

Connections between Eqs. (1.1) and (1.4) are illuminated by the following propositions:

1.1.F. If $D$ is bounded regular domain, $f$ is continuous, and $u = i_D(K_D f)$ is locally bounded, then $u$ is the minimal solution of (1.1) with the boundary value $f$.

1.1.G. If $u \in \mathscr{U}(D)$ and if $D' \Subset D$, then[1]

$$u = \mathscr{E}_{D'}(u) + K_{D'} u. \tag{1.5}$$

Proposition 1.1.G implies that, for every $y \in \mathscr{U}(D)$ and every $D' \Subset D$,

$$i_{D'}(K_{D'} u) \leqslant u. \tag{1.6}$$

[1] Writing $D' \Subset D$ means that $D'$ is a bounded regular open set and that the closure of $D'$ is contained in $D$.

1.2. *Superdiffusion.* Our principal tool is a superdiffusion $X$ in $\mathbb{R}^d$. This is a family of random measures $(X_D, P_\mu)$ where $D$ is an open subset of $\mathbb{R}^d$ and $\mu$ belongs to be set $\mathscr{M}$ of all finite measures on $\mathbb{R}^d$. The probability distribution of $X_D$ relative to $P_\mu$ is determined by the expression

$$P_\mu e^{-\langle f, X_D \rangle} = e^{-\langle v, \mu \rangle} \qquad \text{for} \quad f \in \mathscr{B}_+ \tag{1.7}$$

where

$$v + \mathscr{E}_D(v) = K_D f. \tag{1.8}$$

The joint probability distributions of $X_{D_1}, ..., X_{D_n}$ can be evaluated using the following Markov property. Let $\mathscr{F}_{\subset D}$ and $\mathscr{F}_{\supset Q}$ be the $\sigma$-algebras generated, respectively, by $X'_D$, $D' \subset D$ and $X''_D$, $D'' \supset D$. Then

$$P_\mu\{ Y \mid \mathscr{F}_{\subset D}\} = P_{X_D} Y \qquad P_\mu\text{-a.s..} \tag{1.9}$$

for every $\mathscr{F}_{\supset D}$-measurable function $Y \geqslant 0$ and for all $\mu \in \mathscr{M}$.

Denote by $P_x$ the measure $P_\mu$ corresponding to Dirac's measure $\delta(B) = 1_B(x)$. Put

$$V_D(f)(x) = -\log P_x e^{-\langle f, X_D \rangle}. \tag{1.10}$$

Note that, for every $f \in \mathscr{B}_+$, $v = V_D(f)$ is a solution of Eq. (1.8) which implies that

$$Lv = v^2 \qquad \text{in } D. \tag{1.11}$$

For an arbitrary solution $u \in \mathscr{U}(D)$, we define the *stochastic boundary value* SBV$(u)$ which is a functional of $X$. For every functional $Z$, we define its *potential* PT$(Z)$ and its *log-potential* LPT$(Z)$. We characterize the class $\mathfrak{Z}_1(D) = \{\mathrm{SBV}(u)\colon u \in \mathscr{U}_i(D)\}$ and we prove that SBV is a 1–1 map from $\mathscr{U}_1(D)$ onto $\mathfrak{Z}_1(D)$ and LPT is the inverse map from $\mathfrak{Z}_1(D)$ onto $\mathscr{U}_1(D)$. If $Z = \mathrm{SBV}(u)$ then $h = \mathrm{PT}(Z)$ is the $L$-harmonic function corresponding to $u$ and $v = -\mathrm{LPT}(-Z)$ satisfies Eq. (1.11).

We set

$$W_D(f)(x) = \log P_x e^{\langle f, X_D \rangle}. \tag{1.12}$$

A basis for our investigation is provided by the following theorem proved in Section 2.

THEOREM 1.1. *For every open set $D$ and for every $f \in \mathscr{B}_+$, the function $u = W_D(f)$ is the minimal solution of the equation*

$$u = G_D(u^2) + K_D f. \tag{1.13}$$

(*In other words*, $W_D = i_D K_D$.)

1.3. *Class* $\mathfrak{Z}$. We say that a *sequence $D_n$ exhausts $D$* if $D_1 \Subset D_2 \Subset \cdots \Subset D_n \Subset \cdots$ and the union of $D_n$ coincides with $D$.

Put $\mu \in \mathscr{M}(D)$ if $\mu \in \mathscr{M}$ and $\mu(D^c) = 0$. Denote by $\mathscr{M}_c(D)$ the union of $\mathscr{M}(D')$ over all $D' \Subset D$.

Suppose that

$$Z = \lim \langle u, X_{D_n} \rangle \qquad P_\mu\text{-a.s.} \tag{1.14}$$

for every sequence $D_n$ exhausting $D$ and for all $\mu \in \mathscr{M}_c(D)$. Then we say that $Z$ is the *stochastic boundary value of $u$* and we write $Z = \mathrm{SBV}(u)$. Note that any two versions of $\mathrm{SBV}(u)$ coincide $P_\mu$-a.s. for all $\mu \in \mathscr{M}(D)$.

It is proved in [3] that $\mathrm{SBV}(v)$ exists for every positive solution of Eq. (1.11) and that it belongs to the class $\mathfrak{Z}(D)$ defined by the following conditions:

1.3.A. $Z$ is $\mathscr{F}_{\supset D'}$-measurable for every $D' \Subset D$.

1.3.B. $\log P_\mu e^{-Z} = \int \mu(dx) \log P_x e^{-Z}$ for all $\mu \in \mathscr{M}_c(D)$.

1.3.C. $P_x\{Z < \infty\} > 0$ for all $x \in D$.

Moreover, if $Z \in \mathfrak{Z}(D)$, then

$$v(x) = -\log P_x e^{-Z} \tag{1.15}$$

is a solution of (1.11). Formulae (1.14) and (1.15) establish a 1–1 correspondence between positive solutions of (1.11) and $Z \in \mathfrak{Z}(D)$.

We denote by $\mathfrak{Z}_1(D)$ a subclass of class $\mathfrak{Z}(D)$ determined by the condition

1.3.C*. $P_\mu e^Z < \infty$ for every $\mu$ in $\mathscr{M}_c(D)$.

(Clearly, 1.3.C* implies 1.3.C.)

1.4. *Probabilistic Description of Class* $\mathscr{U}_1(D)$. One of the main results of the present paper is:

THEOREM 1.2. *Stochastic boundary values exists for all $u \in \mathscr{U}(D)$ and for all $h \in \mathscr{H}(D)$. A function $u \in \mathscr{B}_+(D)$ belongs to $\mathscr{U}_1(D)$ if and only if it satisfies the conditions*:

1.4.A. *$W_{D'}(u)=u$ for all $D'\Subset D$.*

1.4.B. *Family $e^{\langle u, X_{D'}\rangle}$, $D'\Subset D$ is uniformly $P_\mu$-integrable for every $\mu\in\mathcal{M}_c(D)$.*

SBV *is a 1–1 mapping from $\mathcal{U}_1(D)$ onto $\mathfrak{Z}_1(D)$ and*

$$\mathrm{LPT}(Z)(x)=\log P_x e^Z.$$

*is the inverse mapping.*

1.5. *Probabilistic Description of Class $\mathcal{H}_1(D)$.*

THEOREM 1.3. *A function $h\in\mathcal{H}(D)$ belongs to $\mathcal{H}_1(D)$ if and only if*

1.5.A. *There exists a locally bounded function $\varphi(x)$ such that $W_{D'}(h)(x)\leqslant\varphi(x)$ for all $D'\Subset D$ and all $x\in D'$.*

SBV *is a 1–1 mapping from $\mathcal{H}_1(D)$ onto $\mathfrak{Z}_1(D)$ and*

$$\mathrm{PT}(Z)(x)=P_x Z$$

*is the inverse mapping. For every $Z\in\mathfrak{Z}_1(D)$,*

$$\mathrm{LPT}(Z)=i_D[\mathrm{PT}(Z)]. \tag{1.16}$$

It is proved in [3] that the class $\mathfrak{Z}(D)$ is convex: if $Z_1, Z_2\in\mathfrak{Z}(D)$, then $\hat{Z}=p_1Z_1+p_2Z_2\in\mathfrak{Z}(D)$ for all $p_1, p_2\geqslant 0$, $p_1+p_2=1$.

By Jensen's inequality,

$$e^{\hat{Z}}\leqslant p_1e^{Z_1}+p_2e^{Z_2},$$

and therefore the classes $\mathfrak{Z}_1(D)$ and $\mathcal{H}_1(D)$ are also convex.

1.6. *Probabolic Setting.* For every open subset $Q$ of $S=\mathbb{R}\times\mathbb{R}^d$, we denote by $\mathbb{C}(Q)$ the set of all continuous functions $u(r,x)$ in $Q$. Put $u\in\mathbb{C}^1(Q)$ if partial derivatives $\dot{u}=\partial u/\partial r$, $\partial u/\partial x_i$ belong to $\mathbb{C}(Q)$; and put $u\in\mathbb{C}^2(Q)$ if, in addition, $\partial^2 u/\partial x_i\,\partial x_j$ are in $\mathbb{C}(Q)$.

We denote by $\mathcal{U}(Q)$ the set of all positive $u\in\mathbb{C}^2(Q)$ such that

$$\dot{u}+Lu=-u^2 \qquad \text{in } Q. \tag{1.17}$$

(From the probabilistic point of view, Eq. (1.17) is more natural then the equation $\dot{v}=LV+v^2$ usually considered by analysts. Both equations can be obtained from each other by the time reversal $v(r,x)=u(-r,x)$.)

We introduce operators

$$K_Q f(r, x) = \Pi_{r,x} f(\tau, \xi_\tau) \tag{1.18}$$

and

$$G_Q f(r, x) = \Pi_{r,x} \int_r^\tau f(s, \xi_s)\, ds, \tag{1.19}$$

where $\tau$ is the first exit time of $(t, \xi_t)$ from $Q$. Let $\mathscr{E}_Q(u) = G_Q(u^2)$. We define a relation $Q' \Subset Q$ and a concept of a sequence $Q_n$ exhausting $Q$ in the same way as in the elliptic setting with the following definition of regularity. Let $\xi = (\xi_t, \Pi_{r,x})$ be a diffusion in $\mathbb{R}^d$ with the generator $L$. A point $(r, x)$ on the boundary $\partial Q$ of $Q$ is called *regular* if, for every $r' > r$, $\Pi_{r,x}\{(t, \xi_t) \in Q$ for all $r < t < r'\} = 0$. A set $Q$ is called regular if the set $\partial_r Q$ of all regular points satisfies the condition

$$\Pi_{r,x}\{\xi_\tau \in \partial_r Q\} = 1 \qquad \text{for all} \quad (r, x) \in Q.$$

(Here $\tau$ is the first exit time from $Q$.) Definitions of operators $i$ and $j$ and classes $\mathscr{U}_1$ and $\mathscr{H}_1$ remain the same with the only difference that $\mathscr{H}(Q)$ means now the set of all positive solutions of the linear parabolic equations $\dot{u} + Lu = 0$. Propositions 1.1.A–1.1.G hold also in a parabolic case.

Now we use a time-inhomogeneous version of a superdiffusion—that is a family of random measures $X = (X_Q, P_\mu)$ where $Q$ are open subsets of $S$ and $\mu$ are finite measures on $S$. We keep the definitions of SBV$(u)$ and of classes $\mathfrak{Z}$ and $\mathfrak{Z}_1$ the same and we put

$$W_Q(f)(r, x) = \log P_{r,x} e^{\langle f, X_Q \rangle}, \qquad \mathrm{LPT}(Z)(r, x) = \log P_{r,x} e^Z,$$
$$\mathrm{PT}(Z)(x) = P_{r,x} Z,$$

where $P_{r,x}$ is the measure $P_\mu$ corresponding to $\delta_{r,x}(B) = 1_B(r, x)$. Theorems 1.1. through 1.3 remain true and their proofs need only minor modifications.

## 2. PROOF OF THEOREM 1.1

2.1. *Cumulants of a Positive Infinitely Divisible Random Variable.* A positive random variable on a measurable space $(\Omega, \mathscr{F})$ is a pair $(Z, P)$ where $Z \in \mathscr{F}$ and $P$ is a probability measure on $\mathscr{F}$.[2]

[2] Writing $Z \in \mathscr{F}$ means that $Z \geqslant 0$ is measurable with respect to a $\sigma$-algebra $\mathscr{F}$.

The Laplace transform of $Z$ has a representation

$$P(e^{-\lambda Z})=\sum_0^n \frac{a_k}{k!}(-\lambda)^k+o(\lambda^n) \qquad \text{as} \quad \lambda\downarrow 0 \tag{2.1}$$

if and only if $P(Z^n)<\infty$. Moreover, $a_k=P(Z^k)$ for $k=1, ..., n$.

The *cumulants* $\sigma_1 \cdots \sigma_n$ of $(Z, P)$ are defined by the formula

$$\sum_1^n \frac{\sigma_k}{k!}\lambda^k=\log\sum_0^n \frac{a_k}{k!}\lambda^k+o(\lambda^n) \qquad \text{as} \quad \lambda\downarrow 0 \tag{2.2}$$

assuming that $a_n=P(Z^n)<\infty$. Formula (2.2) is equivalent to

$$\log Pe^{-\lambda Z}=\sum_1^n \frac{\sigma_k}{k!}(-\lambda)^k+o(\lambda^n) \qquad \text{as} \quad \lambda\downarrow 0. \tag{2.3}$$

(Both (2.2) and (2.3) are equivalent to a set of algebraic relations)

$$\sigma_1=a_1, \qquad \sigma_2=a_2-a_1^2, \qquad \sigma_3=a_3-3a_2a_1+2a_1^3, ...$$

(see, e.g., [6, Chap. 3]). If $a_n<\infty$ and if $\sigma_n\geqslant 0$ for all $n$, then

$$\log Pe^{\lambda Z}=\sum_1^\infty \frac{\sigma_k}{k!}\lambda^k \qquad \text{for all} \quad \lambda\geqslant 0. \tag{2.4}$$

A positive random variable $(Z, P)$ is *infinitely divisible* if, for every $n$, there exists $Z_n$ such that $Pe^{\lambda Z}=(Pe^{\lambda Z_n})^n$ for all $\lambda<0$. This condition implies that, for all $\lambda<0$,

$$\log Pe^{\lambda Z}=m\lambda+\int_0^\infty (e^{\lambda t}-1)\,\ell(dt), \tag{2.5}$$

where $m\geqslant 0$ and $\ell$ is a measure on $R_+$ (the Lévy measure; see, e.g., Theorem 6.1 in [5]). We get from (2.5) the following expressions for the cumulants:

$$\begin{aligned} \sigma_1&=m+\int_0^\infty t\ell(dt), \\ \sigma_n&=\int_0^\infty t^n\ell(dt) \qquad \text{for} \quad n>1. \end{aligned} \tag{2.6}$$

Hence $\sigma_n\geqslant 0$ for all $n$. Formulae (2.6) can be used to define $\sigma_n$ independently of the finiteness of the moments. Note that $a_n<\infty$ if and only if $\sigma_n<\infty$.

Suppose that $a_n<\infty$ for all $n$ and that, for all $\lambda\leqslant 0$,

$$\log Pe^{\lambda Z}=\int \log P_x e^{\lambda Z}\mu(dx), \tag{2.7}$$

where $\mu$ is a finite measure on a measurable space $(E, \mathscr{B})$ and $(Z, P_x)$ is, for every $x\in E$, a positive infinitely divisible random variable. Formula (2.7) implies a relation $\sigma_n=\int_E \sigma_n(x)\,\mu(dx)$ between the cumulants of $(Z, P_\mu)$ and $(Z, P_x)$. Since (2.4) is applicable to $P$ and to $P_x$, we conclude that (2.7) holds for all $\lambda>0$.

2.2. *Proof of Theorem* 1.1 *for bounded D and f.* Let $Z=\langle f, X_D\rangle$. It follows from (1.7) that, for every $\mu\in\mathscr{M}$, the random variable $(Z, P_\mu)$ is infinitely divisible and that, for all $\lambda\leqslant 0$,

$$\log P_\mu e^{\lambda Z}=\int_D \log P_x e^{\lambda Z}\mu(dx). \tag{2.8}$$

Suppose that $D$ and $f$ are bounded and prove, by induction in $n$, the following recursive formulae for cumulants $\sigma_k(x)$ of $(Z, P_x)$:

$$\sigma_1=K_D f, \tag{2.9}$$

$$\sigma_n=\sum_1^{n-1}\binom{n}{k} G_D(\sigma_k\sigma_{n-k}) \qquad \text{for} \quad n>1. \tag{2.10}$$

Denote by $\mathscr{R}_n$ the class of positive Borel functions $R_\lambda(x)$ such that

$$\sup_D R_\lambda(x)=o(\lambda^n) \qquad \text{as} \quad \lambda\downarrow 0.$$

For a bounded domain $D$, $G_D(1)$ is bounded and therefore $G_D$ preserves class $\mathscr{R}_n$.

It follows from (1.7) and (1.8), that, for $\lambda\geqslant 0$,

$$v_\lambda(x)=-\log P_x e^{-\lambda\langle f, X_D\rangle} \tag{2.11}$$

satisfies the equation

$$v_\lambda+G_D(v_\lambda^2)=\lambda K_D f. \tag{2.12}$$

Note that $0\leqslant v_\lambda\leqslant\lambda K_D f\leqslant\lambda\,\|f\|$, where $\|f\|=\sup_D|f(x)|$. Hence, $0\leqslant v_\lambda-\lambda K_D f=G_D(v_\lambda^2)\leqslant\lambda^2\,\|G_D 1\|\,\|f\|^2$ and (2.3) implies (2.9) and

$$v_\lambda=\sigma_1+R_1, \tag{2.13}$$

where $R_1\in\mathscr{R}_1$.

Suppose that (2.10) holds for all $n<N$ and that, for every $n<N$,

$$-v_\lambda=\sum_{k=1}^{n}(-\lambda)^k\frac{\sigma_k}{k!}+R_n, \tag{2.14}$$

where $R_n\in\mathscr{R}_n$.

Then

$$v_\lambda^2=\sum_{k=1}^{N}(-\lambda)^k\sum_{i=1}^{k-1}\frac{\sigma_i}{i!}\frac{\sigma_{k-i}}{(k-i)!}+R'_N$$

with $R'_N\in\mathscr{R}_N$ and, by (2.11) and (2.12),

$$\begin{aligned}\log P_x e^{-\lambda Z}&=-v_\lambda=-\lambda K_D f+G_D(v_\lambda^2)\\&=-\lambda K_D f+\sum_{k=1}^{N}\tilde{\sigma}_k\frac{(-\lambda)^k}{k!}+R''_N,\end{aligned}$$

where

$$\tilde{\sigma}=\sum_{i=1}^{k-1}\binom{k}{i}G_D(\sigma_i\sigma_{k-i})$$

and $R''_N=G_D(R'_N)$ belongs to $\mathscr{R}_N$. By comparing this with (2.3), we conclude that $\tilde{\sigma}_k=\sigma_k$ for all $k\leqslant N$. Therefore (2.10) and (2.14) hold for $N$.

By applying formulae in Section 2.1 to $Z=\langle f, X_D\rangle$, we get:

2.2.A. All the cumulants $\sigma_n(\mu)$ of $(Z, P_\mu)$ are positive.

2.2.B. For all $\lambda\geqslant 0$,

$$\log P_\mu e^{\lambda Z}=\sum_1^\infty\frac{\sigma_k(\mu)}{k!}\lambda^k. \tag{2.15}$$

2.2.C. For all $\mu$,

$$\sigma_n(\mu)=\int\sigma_n(x)\,\mu(dx). \tag{2.16}$$

2.2.D. Formula (2.8) holds for all $\lambda\in\mathbb{R}$.

Equation (1.13) for $u=W_D(f)$ follows from (2.15) and (2.10). Suppose that $v$ is an arbitrary solution of (1.13). By using (2.9) and (2.10), we prove, by induction, that, for all $n$,

$$v\geqslant\sum_1^n\frac{\sigma_k}{k!}$$

and, by (2.15), $v\geqslant u$.

*Remark.* By 2.2.D and (1.12), for every bounded $D$ and $f$,

$$P_\mu e^{\langle f,\, X_D\rangle} = e^{\langle W_D(f),\, \mu\rangle} \tag{2.17}$$

for all $\mu \in \mathcal{M}(D)$, $f \in \mathcal{B}_+$.

2.3. *General Case.* For an arbitrary $f \in \mathcal{B}_+$

$$W_D(K_D f) = \lim_{n\to\infty} W_D(K_D f_n),$$

where $f_n = f \wedge n$. Since $W_D(f_n) = i_D(K_D f_n)$, we have $W_D(f) = i_D(K_D f)$ by 1.1.B.2.

To cover the case of an arbitrary $D$, we use the following properties of superdiffusion:

2.3.A. $$P_\mu \langle f, X_D\rangle = \langle K_D f, \mu\rangle. \tag{2.18}$$

(See, e.g. [2, I, (1.20a)].)

2.3.B. If $D' \subset D$, then $X_{D'}(B) \leqslant X_D(B)$ for all $B \subset \partial D' \cap \partial D$. (See [4, Lemma 3.1].)

Let $\tau$ be the first exit time from $D$ and let $\sigma_n$ be the first exit time from the ball $B_n = \{x\colon |x| < n\}$. Then $\tau_n = \tau \wedge \sigma_n$ is the first exit time from $D_n = D \cap B_n$. Consider $f_n = 1_{B_n} f$. Since $\sigma_n \to \infty$ $\Pi_x$-a.s., we have

$$K_{D_n} f_n(x) = \Pi_x f(\xi_\tau)\, 1_{\tau < \sigma_n} \to \Pi_x f(\xi_\tau) = K_D f(x).$$

It follows from 2.3.B that $Y_n = \langle f_n, X_{D_n}\rangle$ is a monotone increasing sequence and that its limit $Y$ does not exceed $\langle f, X_D\rangle$. By 2.3.A, $P_x Y_n = K_{D_n} f_n$. Hence, $P_x \tilde{Y} = \lim K_{D_n} f_n(x) = K_D f(x) = P_x \langle f, X_D\rangle$ and therefore $\tilde{Y} = \langle f, X_D\rangle$. We conclude that $W_{D_n}(f_n) \uparrow W_D(f)$, and $W_D(f)$ is the minimal solution of (1.13) by 1.1.B.2.

2.4. *A Series for $i_D(h)$.* Note that, for every $h \in \mathcal{H}_1(D)$,

$$i_D(h) = \sum_1^\infty \frac{\sigma_k}{k!}, \tag{2.19}$$

where $\sigma_k$ are defined by the recursive formula (2.10) with $\sigma_1 = h$. Indeed, the computations in Section 2.2 show that the function on the right side of (2.19) satisfies (1.13) and that it does not exceed any other solution of (1.13).

## 3. PROOF OF THEOREMS 1.2 AND 1.3

3.1. *Stochastic Boundary Values.* Let $\mu \in \mathcal{M}_c(D)$ and let $\{D_n\}$ exhaust $D$. If $u \in \mathcal{U}(D)$, then $Y_n = e^{\langle u, X_{D_n} \rangle}$ is a supermartingale relative to $(\mathcal{F}_{\subset D_n}, P_\mu)$. Indeed, by Theorem 1.1 and (1.6), $W_{D_n}(u) = i_{D_n}[K_{D_n}(u)] \leqslant u$. By (2.17),

$$P_\mu Y_n = e^{\langle W_{D_n}(u), \mu \rangle} \leqslant e^{\langle u, \mu \rangle} < \infty. \tag{3.1}$$

By (1.9) and (3.1),

$$P_\mu\{Y_{n+1} \mid \mathcal{F}_{\subset D_n}\} = P_{X_{D_n}} Y_{n+1} \leqslant e^{\langle u, X_{D_n} \rangle} = Y_n \qquad P_\mu\text{-a.s.}$$

Analogously, if $h \in \mathcal{H}(D)$, then $F_n = \langle h, X_{D_n} \rangle$ is a martingale relative to $(\mathcal{F}_{\subset D_n}, P_\mu)$. Therefore stochastic boundary values are defined for all $u \in \mathcal{U}(D)$ and for all $h \in \mathcal{H}(D)$.

Suppose that $u \in \mathcal{U}(D)$ and $h \in \mathcal{H}(D)$ are connected by (1.4). Then,

$$\mathrm{SBV}(u) = \mathrm{SBV}(h). \tag{3.2}$$

Indeed, by 2.3.A,

$$\begin{aligned} P_\mu \langle \mathcal{E}_D u, X_{D_n} \rangle &= \langle K_{D_n} \mathcal{E}_D(u), \mu \rangle \\ &= \Pi_\mu \int_{\tau_n}^{\tau} u(\xi_s)^2 \, ds \to 0 \qquad \text{as} \quad n \to \infty \end{aligned}$$

for every $\mu \in \mathcal{M}_c(D)$ because the first exit time $\tau_n$ from $D_n$ tends to the first exit time $\tau$ from $D$ and

$$\Pi_\mu \int_0^\tau u(\xi_s)^2 \, ds = \langle \mathcal{E}_D(u), \mu \rangle \leqslant \langle u, \mu \rangle < \infty.$$

3.2. *Probabilistic Representation of $u \in \mathcal{U}(D)$.* Note that $h$ belongs to $\mathcal{H}(D)$ if and only if

$$K_{D'} h = h \qquad \text{for all} \quad D' \Subset D. \tag{3.3}$$

Let $h \in \mathcal{H}(D)$. For every $D' \Subset D$, by Theorem 1.1 and (3.3),

$$W_{D'}(h) = i_{D'}(K_{D'} h) = i_{D'}(h) \tag{3.4}$$

and 1.1.B.2 implies that, for every sequence $D_n$ exhausting $D$,

$$i_D(h) = \lim W_{D_n}(h). \tag{3.5}$$

LEMMA 3.1. *Suppose* $\mu \in \mathscr{M}_c(D)$, $u \in \mathscr{U}(D)$, $h=j_D(u)$, *and* $Z=\mathrm{SBV}(h)$. *Then*

$$P_\mu e^Z = e^{\langle i_D(h),\, \mu\rangle}. \tag{3.6}$$

*Proof.* By 1.1.D, $i_D(h) \leqslant u$. Choose $0<\lambda<1$ and put $p=1/\lambda$, $Y_n = e^{\lambda\langle h,\, X_{D_n}\rangle}$. By (3.5), (2.17), and (1.4),

$$P_\mu Y_n^p = P_\mu e^{\langle h,\, X_{Dn}\rangle} = e^{\langle W_{D_n}(h),\, \mu\rangle} \to e^{\langle i_D(h),\, \mu\rangle} \leqslant e^{\langle u,\, \mu\rangle} < \infty. \tag{3.7}$$

Since every bounded set in $L^p$ is uniformly integrable, (3.7) implies

$$\lim P_\mu Y_n = P_\mu \lim Y_n.$$

By (2.17) and (3.5),

$$P_\mu Y_n = e^{\langle W_{D_n}(\lambda h),\, \mu\rangle} \to e^{\langle i_D(\lambda h),\, \mu\rangle}$$

and, since $Y_n \to e^{\lambda Z}$, we get

$$e^{\langle I_D(\lambda h),\, \mu\rangle} = P_\mu e^{\lambda Z}.$$

By passing to the limit as $\lambda \uparrow 1$ and by using 1.1.B.2, we get (3.6). ∎

LEMMA 3.2. *If* $u \in \mathscr{U}_1(D)$, *then* $Z=\mathrm{SBV}(u) \in \mathfrak{Z}_1(D)$ *and* $u=\mathrm{LPT}(Z)$.

*Proof.* Since $i_D[\,j_D(u)]=u$ and, by (3.2), $\mathrm{SBV}[\,j_D(u)]=Z$, we get from (3.6) that

$$\log P_\mu e^Z = \langle u, \mu\rangle \tag{3.8}$$

and $\mathrm{LPT}(Z)=u$.

It follows from (3.8) that $Z$ satisfies 1.3.B and 1.3.C*. Clearly it also satisfies 1.3.A. Hence, $Z \in \mathfrak{Z}_1(D)$. ∎

3.3. *Properties of Log-Potentials and Potentials.* Suppose that $u = \mathrm{LPT}(Z)$ for $Z \in \mathfrak{Z}_1(D)$. It follows from 1.3.C* and 2.2.D that $u$ is locally bounded. Formula 1.3.B implies that

$$P_\mu Z = \int_D P_x Z \mu(dx). \tag{3.9}$$

LEMMA 3.3. *Let* $Z \in \mathfrak{Z}_1(D)$, $u=\mathrm{LPT}(Z)$, $h=\mathrm{PT}(Z)$. *Then for every* $\mu \in \mathscr{M}_c(D)$ *and every* $D' \Subset D$,

$$P_\mu\{e^Z \mid \mathscr{F}_{\subset D'}\} = e^{\langle u,\, X_{D'}\rangle} \tag{3.10}$$

*and*

$$P_\mu\{Z \mid \mathcal{F}_{D'}\} = \langle h, X_{D'}\rangle. \tag{3.11}$$

*Function $u$ satisfies conditions* 1.4.A *and* 1.4.B.

*Proof.* For every $D' \Subset D$, $P_\mu\{X_{D'} \in \mathcal{M}_c(D)\} = 1$ and, by 1.3.A and (1.9),

$$\begin{aligned} P_\mu\{e^Z \mid \mathcal{F}_{\subset D'}\} &= P_{X_{D'}} e^Z, \\ P_\mu\{Z \mid \mathcal{F}_{\subset D'}\} &= P_{X_{D'}} Z. \end{aligned} \tag{3.12}$$

By 2.2.D, (3.9), and (1.12), this implies (3.10) and (3.11).

It follows from (3.10) that $e^{u(x)} = P_x e^Z = P_x e^{\langle u, XD'\rangle} = e^{W_D(x)}$. Hence, $u$ satisfies 1.4.A. Condition 1.4B also follows from (3.10). ▮

LEMMA 3.4. *If $u$ satisfies conditions* 1.4.A *and* 1.4.B, *then* $u \in \mathcal{U}_1(D)$.

*Proof.* By 1.4.A and (2.17), $e^{\langle u, \mu\rangle} = e^{\langle W_{D'}(u), \mu\rangle} = P_\mu e^{\langle u, X_{D'}\rangle}$ for all $D' \Subset D$ and all $\mu \in \mathcal{M}_c(D)$. By 1.4.B, this expression is finite, which implies that $u$ is locally bounded. By Theorem 1.1 and 1.1.G, $i_{D'}(K_{D'}u) = W_{D'}(u) = u$ satisfies the equation $u = \mathcal{E}_{D'}(u) + K_{D'}u$. By 1.1.F, $u \in \mathcal{U}(D')$. Hence, $u \in \mathcal{U}(D)$.

Let $D_n$ exhaust $D$. By passing to the limit in the equation $u = \mathcal{E}_{D_n}(u) + K_{D_n}u$, we get $u = \mathcal{E}_D(u) + h$, where $h = \lim K_{D_n}u$. By Theorem 4.1 in [1], $h = j_D(u)$. By (3.2), $Z = \mathrm{SBV}(u) = \mathrm{SBV}(h)$. By passing to the limit in the equation $e^{\langle u, \mu\rangle} = P_\mu e^{\langle u, X_{D_n}\rangle}$, we get $e^{\langle u, \mu\rangle} = P_\mu e^Z$. Hence, $u = \mathrm{LPT}(Z)$. By 1.4.B and (3.5) $u(x) = \log P_x e^Z = \lim \log P_x e^{\langle h, X_{D_n}\rangle} = \lim W_{D_n}(h) = i_D(h) = i_D[j_D(u)]$, and $u \in \mathcal{U}_1(D)$. ▮

LEMMA 3.5. *Function $h \in \mathcal{H}(D)$ belongs to $\mathcal{H}_1(D)$ if and only if it satisfies condition* 1.5.A.

*Proof.* Let $h \in \mathcal{H}(D)$. By (3.4) and 1.1.B.1, $W_{D'}(h) = i_{D'}(h) \leqslant i_D(h)$ and therefore 1.5.A holds for every $h \in \mathcal{H}_1(D)$. On the other hand, by (3.5), 1.5.A implies $i_D(h) \leqslant \varphi$ and therefore $h \in \mathcal{H}_1(D)$.

LEMMA 3.6. *If $Z \in \mathfrak{Z}_1(D)$, then $h = \mathrm{PT}(Z) \in \mathcal{H}_1(D)$, $u = \mathrm{LPT}(Z) \in \mathcal{U}_1(D)$. We have $i_D(h) = u$ and* $\mathrm{SBV}(h) = \mathrm{SBV}(u) = Z$.

*Proof.* It follows from Lemma 3.3 and Lemma 3.4 that $u \in \mathcal{U}_1(D)$.

By (1.9),

$$P_x P_{X_{D'}} Z = P_x Z = h(x).$$

By (3.9), $P_{X_{D'}} Z = \langle h, X_{D'}\rangle$ and, by 2.3.A, $P_x\langle h, X_{D'}\rangle = K_{D'}h(x)$. Hence, $K_{D'}h = h$ for all $D' \Subset D$ and $h \in \mathcal{H}(D)$.

By (3.11),

$$\langle h, X_{D_n} \rangle = P_\mu\{Z \mid \mathscr{F}_{\subset D_n}\} \to Z \qquad P_\mu\text{-a.s.} \tag{3.13}$$

Hence, $\mathrm{SBV}(h) = Z$. By (3.5) and 1.4.B,

$$i_D(h) = \lim \log P_x e^{\langle h, X_{D_n} \rangle} = \log P_x e^Z = u(x). \tag{3.14}$$

Since $u$ is locally bounded, $h \in \mathscr{H}_1(D)$. The last statement of the lemma follows from (3.2). ∎

3.4. *Proof of Theorem* 1.2 *and* 1.3. The existence of stochastic boundary values of $u$ and $h$ is proved in Section 3.1. The characterization of $\mathscr{U}_1(D)$ by conditions 1.4.A and 1.4.B follows from Lemma 3.3 and 3.4. By Lemma 3.2, SBV: $\mathscr{U}_1(D) \to \mathfrak{Z}_1(D)$ and $\mathrm{LPT}[\mathrm{SBV}(u)] = u$ for all $u \in \mathscr{U}_1(D)$. Finally, by Lemma 3.6, LPT: $\mathfrak{Z}_1(D) \to \mathscr{U}_1(D)$ and $\mathrm{SBV}[\mathrm{LPT}(Z)] = Z$ for all $Z \in \mathfrak{Z}_1(D)$.

The characterization of $\mathscr{H}_1(D)$ by condition 1.5.A was proved in Lemma 3.5. By Lemma 3.6, PT: $\mathfrak{Z}_1(D) \to \mathscr{H}_1(D)$ and, if $Z \in \mathfrak{Z}_1(D)$, then $\mathrm{SBV}[\mathrm{PT}(Z)] = Z$. Formula (1.16) also follows from Lemma 3.6. It remains to be proven that, if $h \in \mathscr{H}_1(D)$ and if $Z = \mathrm{SBV}(h)$, then $h' = \mathrm{PT}(Z)$ coincides with $h$. By Lemma 3.6, $h'$ and $h$ have the same stochastic boundary value. By (3.2), the same is true for $i_D(h')$ and $i_D(h)$. By Theorem 1.2, $i_D(h') = i_D(h)$, and by 1.1.C, $h' = h$.

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